On our first day with ellipses, the center was always $(0, 0)$, so our equations were

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

[Horizontal major axis]

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

[Vertical major axis]

The number under x^2 is always $(\text{horizontal radius})^2$
The number under y^2 is always $(\text{vertical radius})^2$

$\boldsymbol{a^2} > \boldsymbol{b^2}$ (The two numbers will never be equal, so make $\boldsymbol{a^2}$ = larger number)

The foci are always on the major axis and are the same distance away from the center of the ellipse (we call that distance $\boldsymbol{c}$).

$$\boldsymbol{c^2} = \boldsymbol{a^2} - \boldsymbol{b^2}$$

Sum of the focal radii = $2\boldsymbol{a}$
($\boldsymbol{a}$ will always be the distance of a radius that contains a focus… and there are two of them!)

Now we are ready to look at ellipses with centers that may not be $(0, 0)$ …

Equation of ellipse with center (h, k)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
[Horizontal major axis]

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$
[Vertical major axis]

The number under x^2 is always (horizontal radius) ^{2}
The number under y^2 is always (vertical radius) ^{2}

In fact, everything we reviewed on the first page is still true here!

Let's see some examples…

Example 1 - Give the center and foci of the ellipse.

$$\frac{(x-3)^2}{144}+\frac{(y+1)^2}{169}=1$$

Find the center

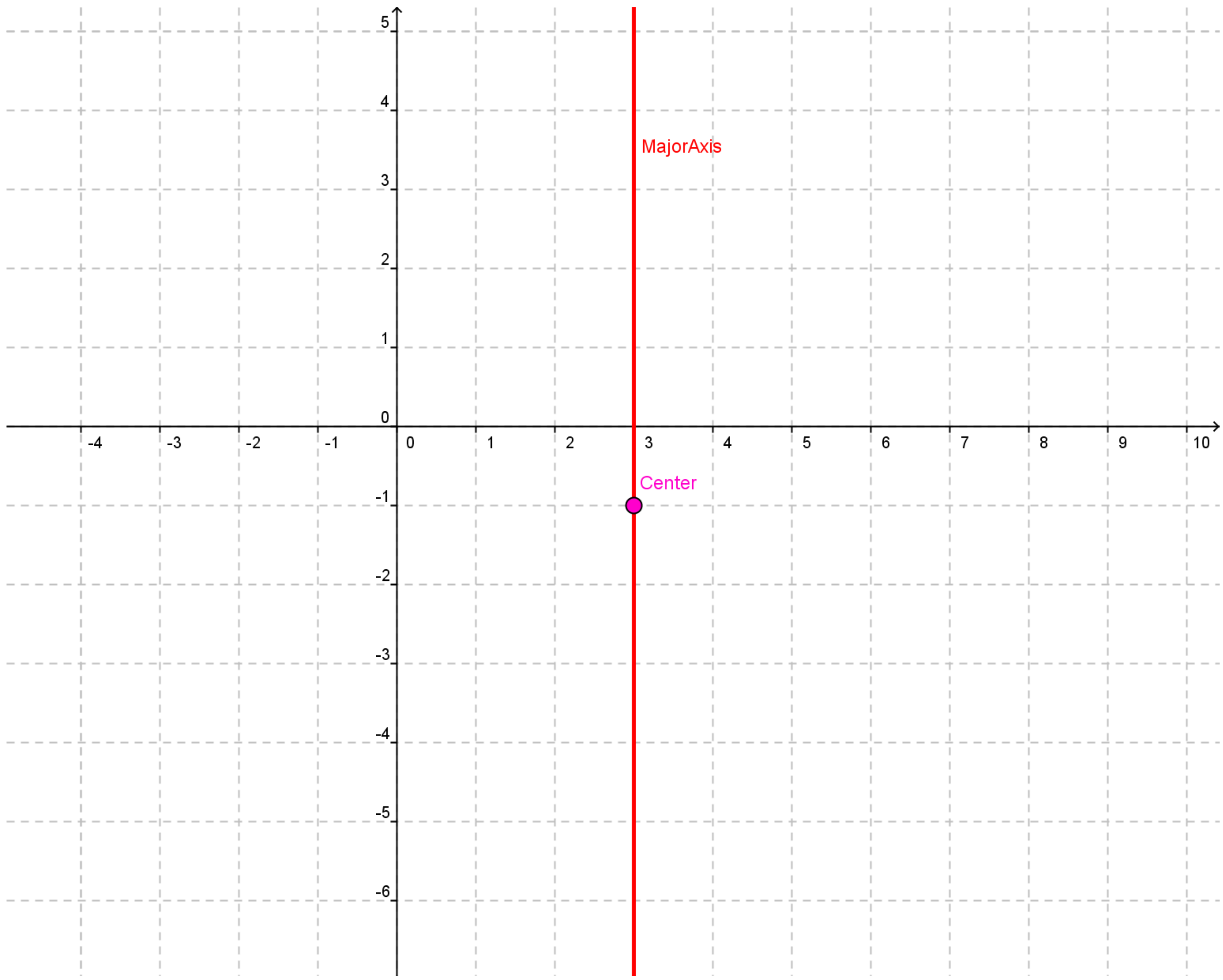
Look for $(\boldsymbol{h},\boldsymbol{k})$ - we've had a lot of practice at this!

$(x-3)^2$ means $\boldsymbol{h}=\mathbf{3}$
$(y+1)^2$ means $\boldsymbol{k}=\mathbf{-1}$ … the center is $(\mathbf{3},\mathbf{-1})$

Find the foci

Step 1– Graph the center and determine direction of major axis

Because the bigger number is under a y^2–term, the direction of the major axis is VERTICAL.



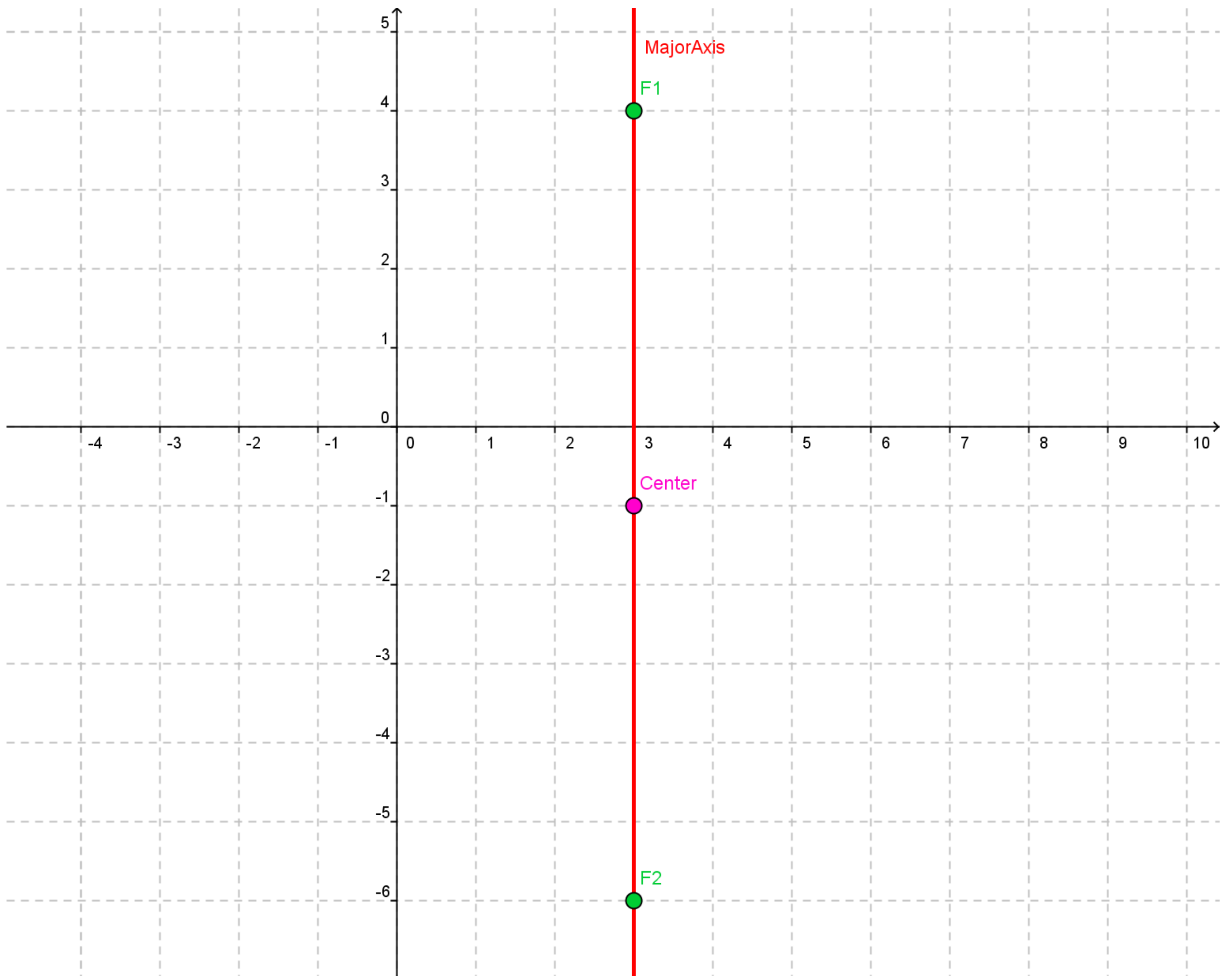
Step 2 - Find c

Remember that $c^2 = a^2 - b^2$. Look at the bottom of the fractions to find a^2 & b^2. The bigger number is 169, so…

$$a^2 = 169 \text{ and } b^2 = 144$$
$$c^2 = 169 - 144 = 25$$
$$c^2 = 25$$
$$c = 5$$

Step 3 – Graph the foci using the information from Steps 1 and 2.

Because the major axis is VERTICAL, we must go UP and DOWN from the center by an amount of 5 (because $c = 5$).



The foci are $(3, 4)$ & $(3, -6)$

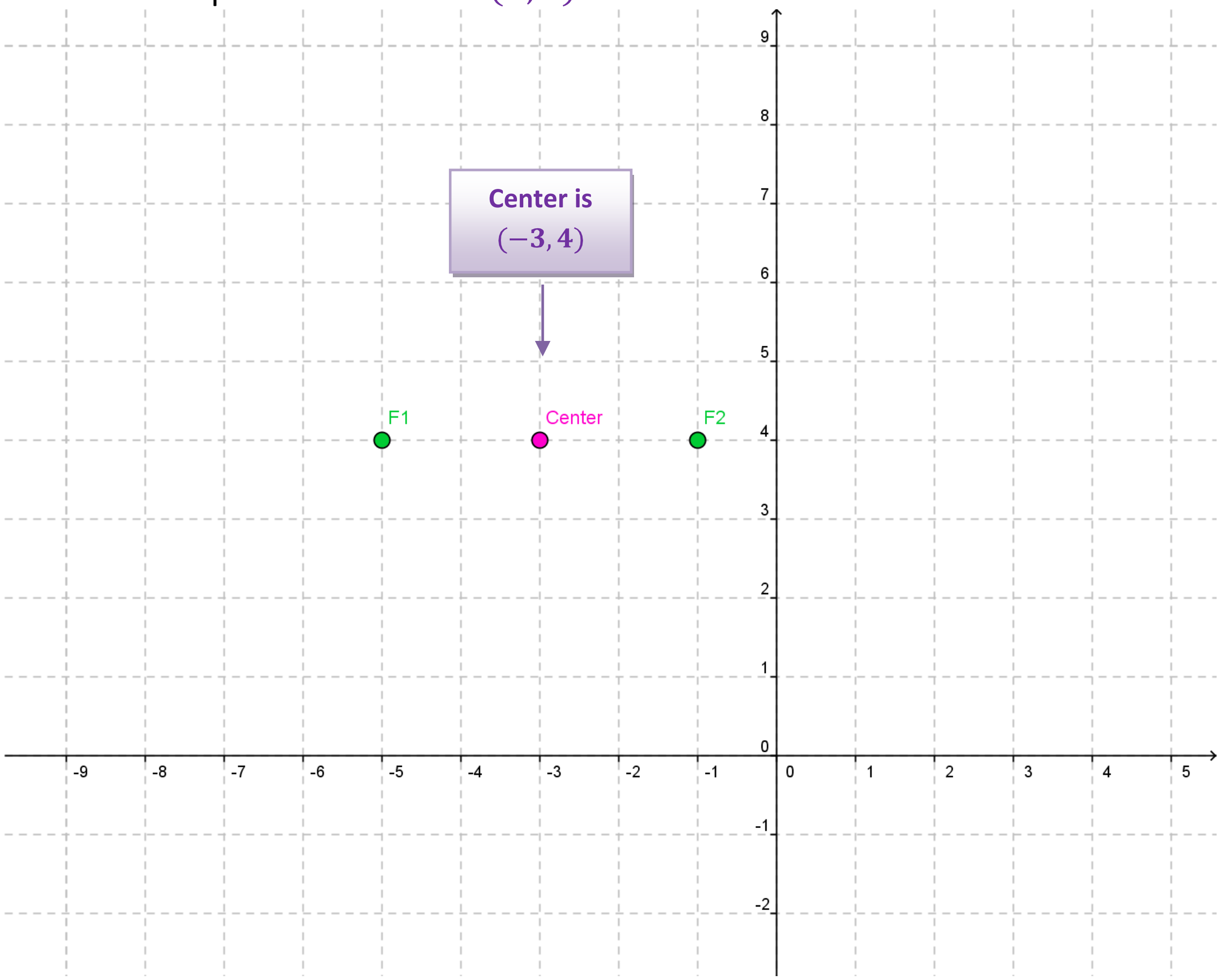
Do you see how this is like $(3, -1 \pm 5)$?

Example 2 - Find an equation of an ellipse with the given information.

Foci: $(-1,\ 4), (-5,\ 4)$
Sum of focal radii: 6

Step 1- Find the center (h, k)

Graph the foci and determine the midpoint of the segment between the foci. That midpoint is the center (h, k) .

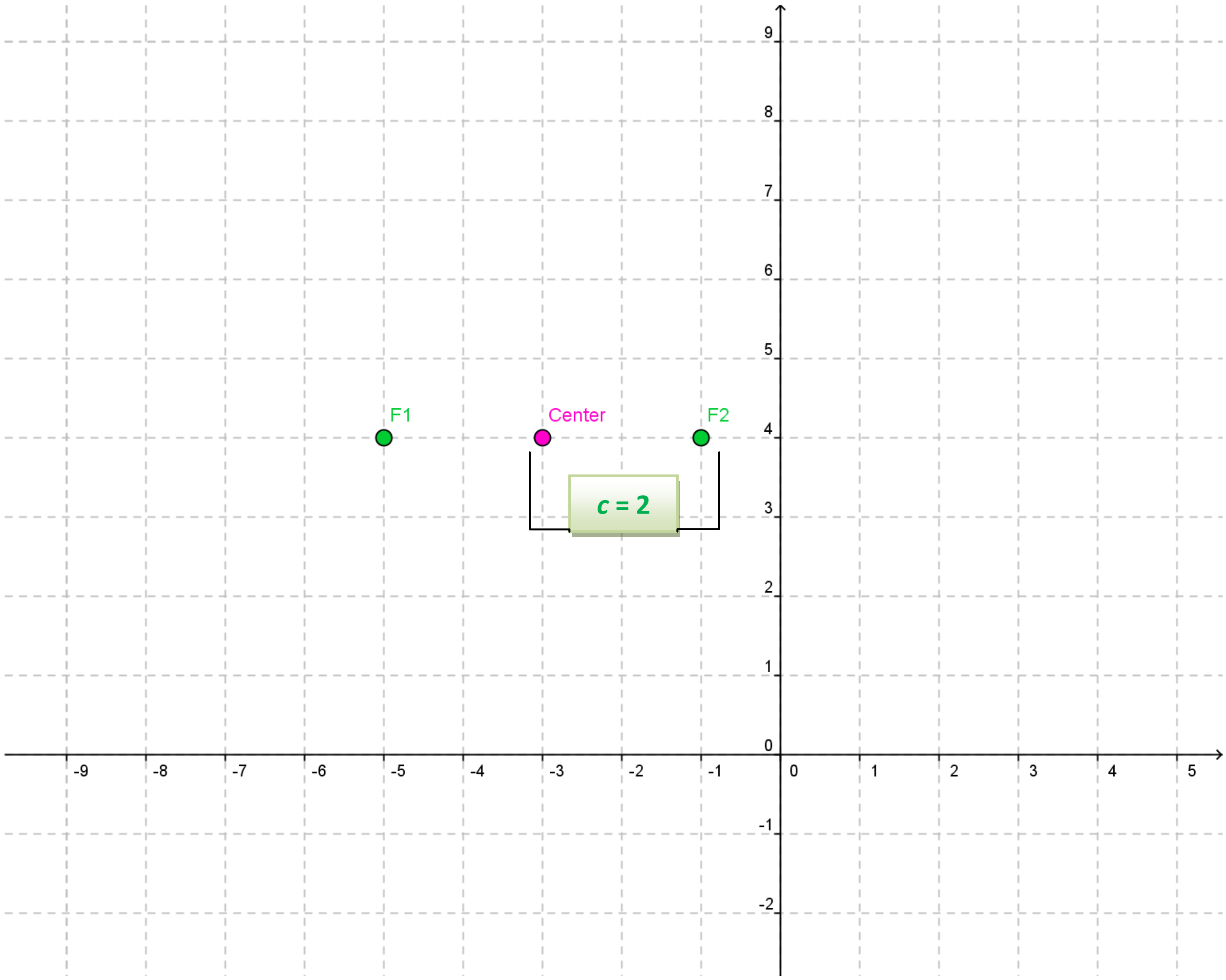


Step 2 – Start writing an ellipse equation

$$\frac{(x-h)^2}{??^2} + \frac{(y-k)^2}{??^2} = 1$$

$$\frac{(x+3)^2}{??^2} + \frac{(y-4)^2}{??^2} = 1$$

Step 3 – Look at the graph to determine c



Step 4 – Calculate a^2

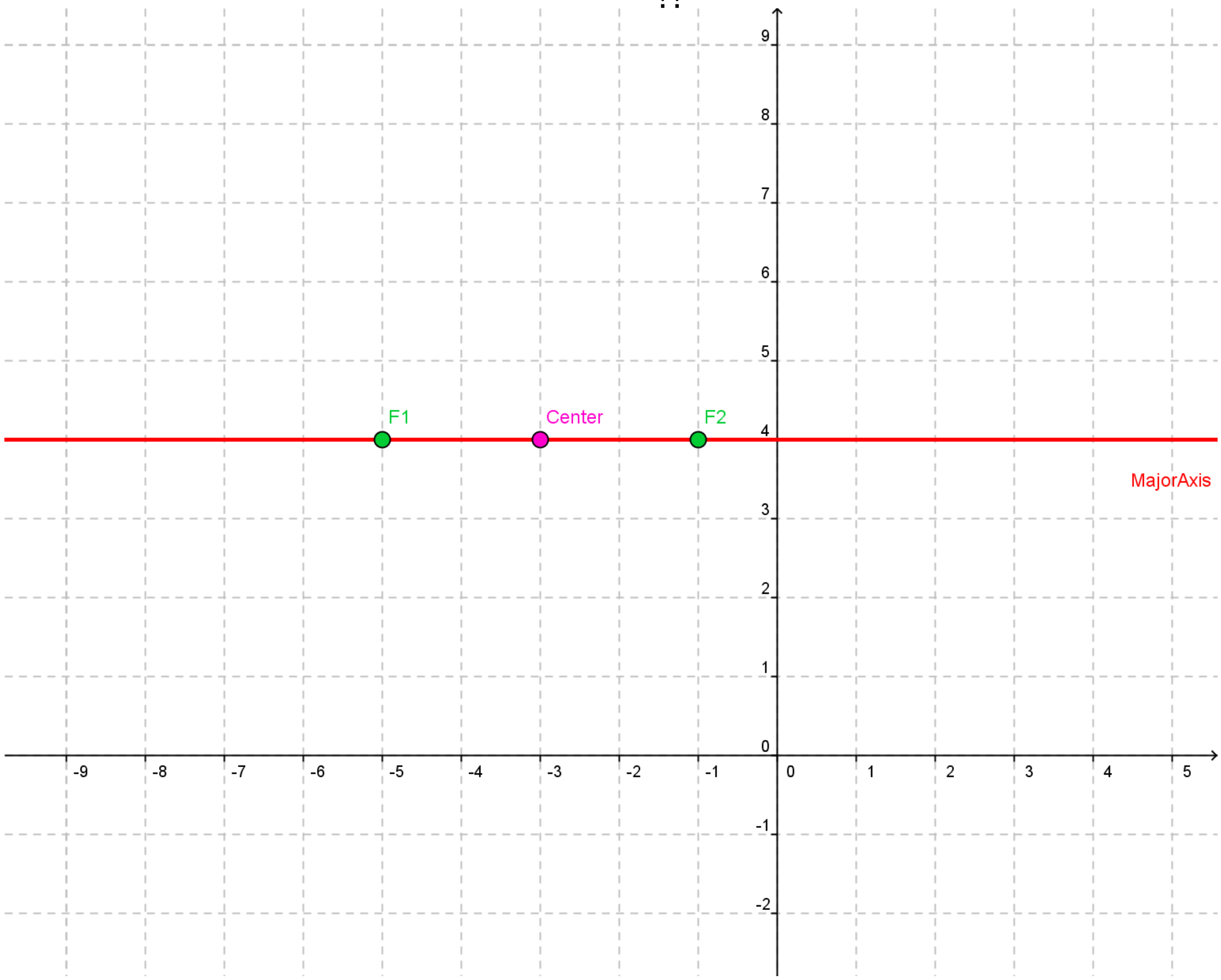
We know the sum of the focal radii = $2a$.

$$2a = 6$$
$$a = 3$$
$$a^2 = 9$$

But does the **9** go under the $\frac{(x+3)^2}{??^2}$ or under the $\frac{(y-4)^2}{??^2}$?

Step 5 – Look at the graph to determine where to put a^2

The center and the foci always lie on the major axis. Our major axis is HORIZONTAL, so we will put a^2 under the $\frac{(x+3)^2}{??^2}$.



$$\frac{(x+3)^2}{9} + \frac{(y-4)^2}{b^2} = 1$$

Step 6 – Calculate b^2

Remember that $c^2 = a^2 - b^2$.

$$c^2 = a^2 - b^2$$
$$2^2 = 9 - b^2$$
$$4 = 9 - b^2$$
$$b^2 = 5$$

$$\frac{(x+3)^2}{9} + \frac{(y-4)^2}{5} = 1$$

<u>Example 3</u> – Rewrite the following equation in the standard form for an ellipse. Find the center, foci, verticies, co-verticies, and direction of major axis of the ellipses. Draw the graph.

$$9x^2 + 16y^2 + 108x - 128y + 436 = 0$$

We can use completing the square to make it look like the ellipse equation:

$$\frac{(x-h)^2}{??^2} + \frac{(y-k)^2}{??^2} = 1$$

Step 1 – Move the free number to the right and put the x^2 & x terms in one group on the left and the y^2 & y terms in another group on the left.

$$9x^2 + 16y^2 + 108x - 128y + 436 = 0$$
$$9x^2 + 108x + 16y^2 - 128y = -436$$

Step 2 – Take out the GCF (greatest common factor) for the x^2 & x terms and the y^2 & y terms.

$$9x^2 + 108x + 16y^2 - 128y = -436$$
$$9(x^2 + 12x \quad) + 16(y^2 - 8y \quad) = -436$$

Step 3 – Complete the square for both groups on the left and remember to carefully balance on the right!

$$9(x^2 + 12x + \quad) + 16(y^2 - 8y + \quad) = -436$$
$$9(x^2 + 12x + 36) + 16(y^2 - 8y + 16) = -436 + 324 + 256$$
$$9(x+6)^2 + 16(y-4)^2 = 144$$

Step 4 – Divide everything by the number on the right side.

$$\frac{9(x+6)^2}{144} + \frac{16(y-4)^2}{144} = \frac{144}{144}$$

$$\frac{(x+6)^2}{16} + \frac{(y-4)^2}{9} = 1$$

← Standard form for an ellipse equation!

Step 5 – Graph center, verticies, and co-verticies.

$$\frac{(x+6)^2}{16} + \frac{(y-4)^2}{9} = 1$$

Use (h, k) to find center: $(-6, 4)$

Use a^2 to find verticies and major axis direction: $a^2 = 16$

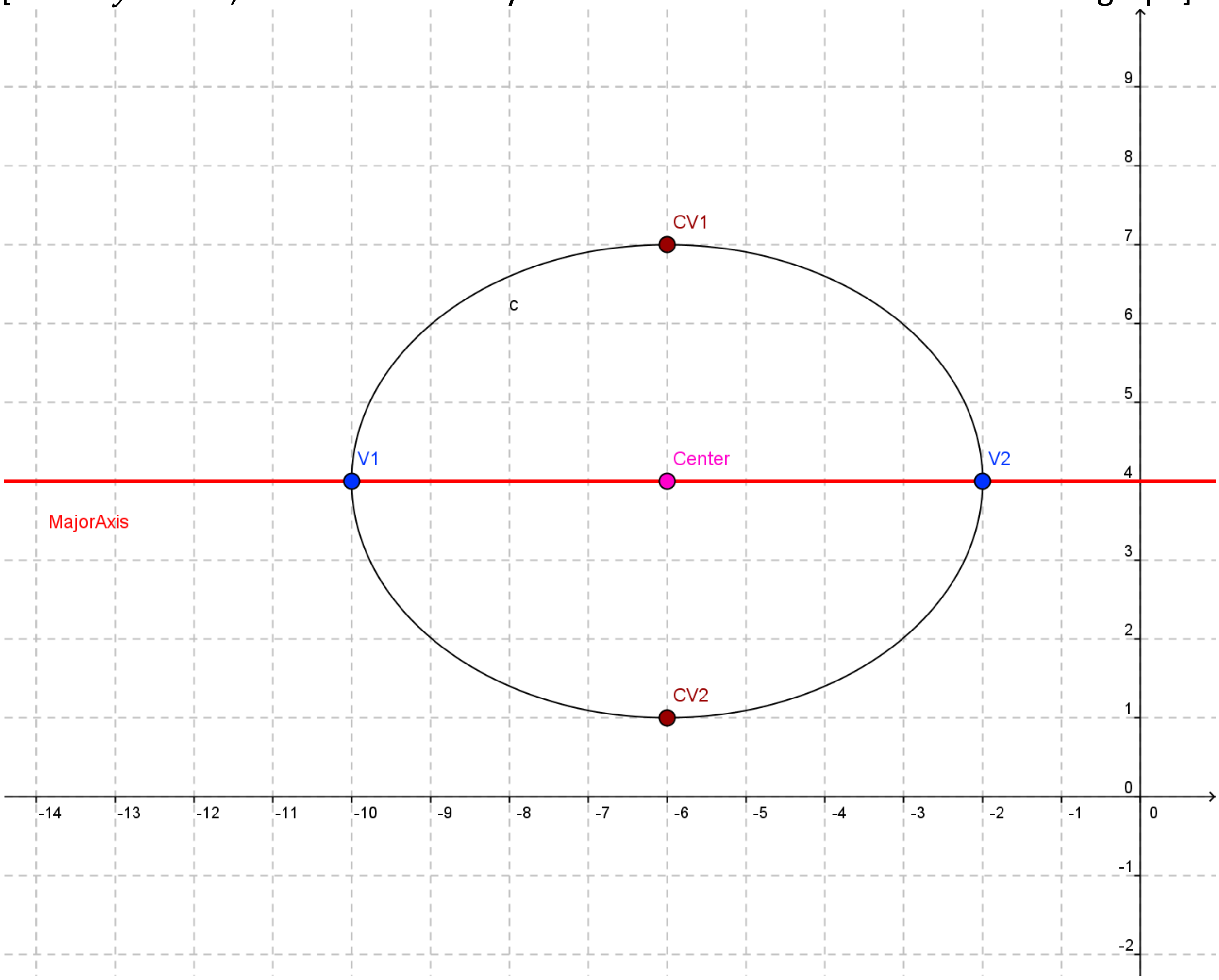
$a = 4$

[Under x^2 term, so must be 4 away from center in HORIZONTAL directions on graph as major axis is HORIZONTAL]

Use b^2 to find co-verticies: $b^2 = 9$

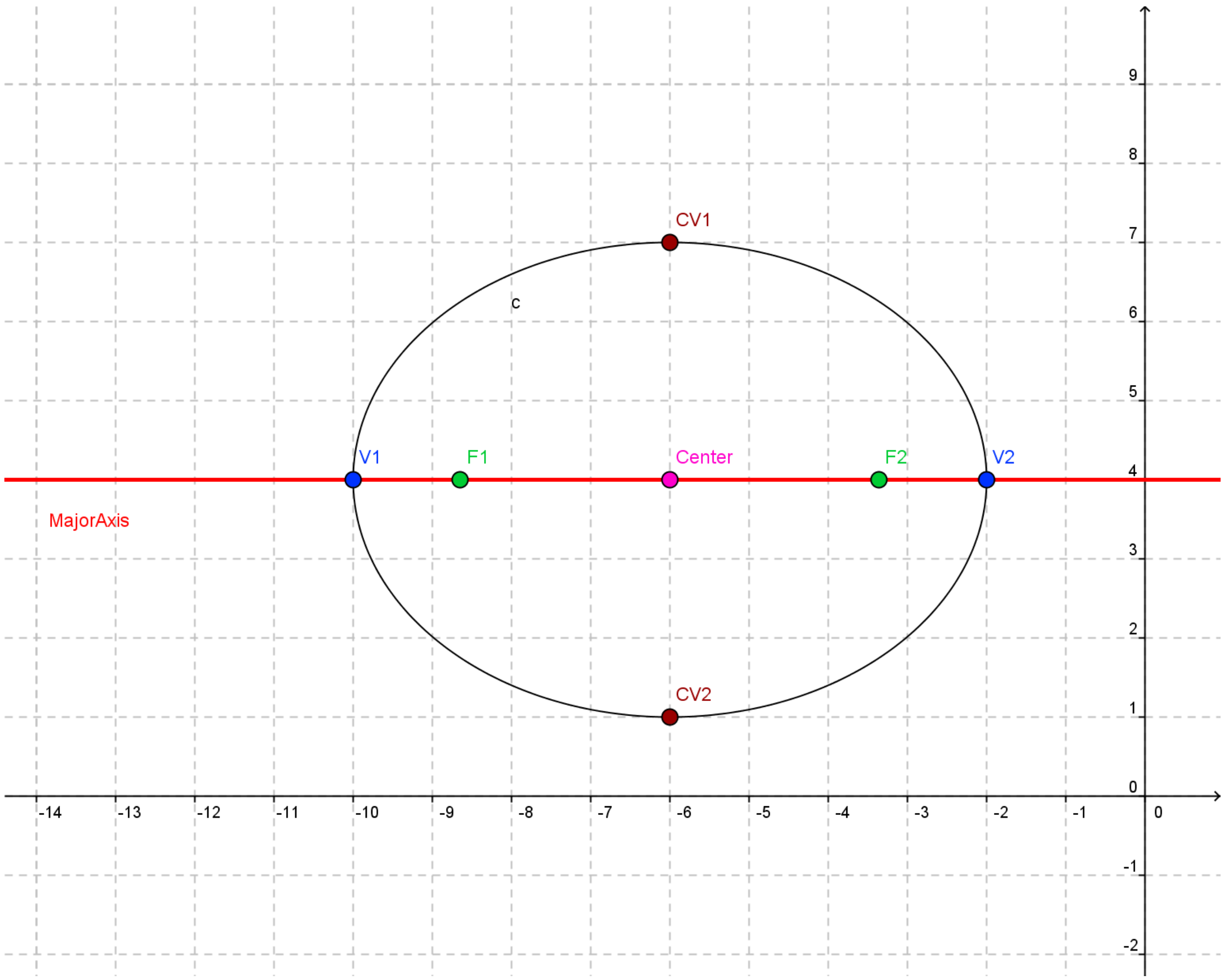
$b = 3$

[Under y^2 term, so must be 3 away from center in VERTICAL directions on graph]



Step 6 – Find (and graph) the foci.

$$c^2 = a^2 - b^2$$
$$c^2 = 16 - 9$$
$$c^2 = 7$$
$$c = \pm\sqrt{7}$$



Center: $(-6, 4)$

Major Axis: Horizontal

Verticies: $(-10, 4)$ & $(-2, 4)$

Co-Verticies: $(-6, 7)$ & $(-6, 1)$

Foci: $(-6 \pm \sqrt{7}, 4)$

May need to use a calculator to estimate where to place foci