Sometimes we can find the sum of an infinite number of terms in a geometric series.

Look at this series: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$

Add first 1 term: $\frac{1}{2} \quad = 0.5$

Add first 2 terms: $\frac{1}{2}+\frac{1}{4} \quad = 0.75$

Add first 3 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \quad = 0.875$

Add first 4 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} \quad = 0.9375$

Add first 5 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} \quad = 0.96875$

Add first 6 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64} \quad = 0.984375$

Add first 7 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128} \quad = 0.9921875$

Add first 8 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256} \quad = 0.99609375$

Add first 9 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512} \quad = 0.998046875$

Add first 10 terms: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024} \quad = 0.9990234375$

Every time we add another term, the sum of the series gets closer to a certain number (in this case the number is 1). The sum never exactly gets to 1, but it gets so very close that we might as well say that the sum is 1.

The amount we add each time is getting closer and closer to zero- it’s almost like we aren’t adding anything at all… we’ve reached a limit.

Sometimes we can't find the sum of an infinite number of terms in a geometric series.

Look at this series: $3 + 12 + 48 + 192 + 768 + \cdots$

Add first 1 term: $3 \quad = 3$

Add first 2 terms: $3 + 12 \quad = 15$

Add first 3 terms: $3 + 12 + 48 \quad = 63$

Add first 4 terms: $3 + 12 + 48 + 192 \quad = 255$

Add first 5 terms: $3 + 12 + 48 + 192 + 768 \quad = 1023$

Add first 6 terms: $3 + 12 + 48 + 192 + 768 + 3072 \quad = 4095$

Add first 7 terms: $3 + 12 + 48 + 192 + 768 + 3072 + 12{,}288 \quad = 16{,}383$

This sum isn't getting closer to anything besides infinity! We keep adding bigger and bigger numbers (not like the first example where we kept adding smaller and smaller numbers until we were practically adding zero).

How can we tell if an infinite geometric series has a sum?

We call the number by which we are multiplying to get consecutive terms the ratio and we let $r = \text{ratio}$.

Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite series DOES HAVE A SUM, and the sum is…

$$S = \frac{a_1}{1 - r}$$

1st term

Ratio

If $|r| \geq 1$, then the infinite series doesn't have a sum.

Example 1- Does the following infinite geometric series have a sum? If so, what is the sum?

$$\sum_{n=1}^{\infty} 5(0.8)^{n-1}$$

The $\text{ratio} = 0.8$, which means there is a sum we can find!
The 1^{st} term $a_1 = 5(0.8)^{1-1} = 5(0.8)^0 = 5(1) = 5$

Now we can use our formula:

$$S = \frac{a_1}{1 - r}$$

$$S = \frac{5}{1 - 0.8} = \frac{5}{.2} = 25$$

The sum of this infinite geometric series is **25**

Example 2- Does the following infinite geometric series have a sum? If so, what is the sum?

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}+\cdots$$

The ratio $= -\frac{1}{3}$, which means there is a sum we can find!
The 1st term $a_1 = 1$

Now we can use our formula:

$$S=\frac{a_1}{1-r}$$

$$S=\frac{1}{1-(-{}^{1}/_{3})}=\frac{1}{1+{}^{1}/_{3}}=\frac{1}{{}^{4}/_{3}}=\frac{3}{4}$$

The sum of this infinite geometric series is $\mathbf{{}^{3}/_{4}}$

Example 3- Does the following infinite geometric series have a sum? If so, what is the sum?

$$\sum_{n=1}^{\infty}10\left(\frac{5}{4}\right)^{n-1}$$

The ratio $= \frac{5}{4}$, which means there isn't a sum we can find!